

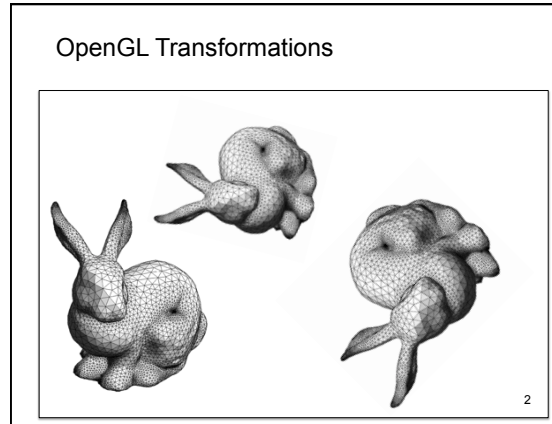
CSCI 480 Computer Graphics
Lecture 4

Transformations

Vector Spaces
Euclidean Spaces
Frames
Homogeneous Coordinates
Transformation Matrices
[Angel, Ch. 4]

January 28, 2013
Jernej Barbic
University of Southern California
<http://www-bcf.usc.edu/~jbarbic/cs480-s13/>

1



OpenGL Transformation Matrices

- Model-view matrix (4x4 matrix)
- Projection matrix (4x4 matrix)

vertices in canonical 3D world coordinate system

vertices in 3D → **Model-view** → **Projection** → vertices in 2D

3

4x4 Model-view Matrix (this lecture)

- Translate, rotate, scale objects
- Position the camera

vertices in canonical 3D world coordinate system

vertices in 3D → **Model-view** → **Projection** → vertices in 2D

4

4x4 Projection Matrix (next lecture)

- Project from 3D to 2D

vertices in canonical 3D world coordinate system

vertices in 3D → **Model-view** → **Projection** → vertices in 2D

5

OpenGL Transformation Matrices

→ **Model-view** ↔ **Projection** →

- Manipulated separately in OpenGL (must set matrix mode):

```
glMatrixMode (GL_MODELVIEW);
glMatrixMode (GL_PROJECTION);
```

6

Setting the Current Model-view Matrix

- Load or post-multiply

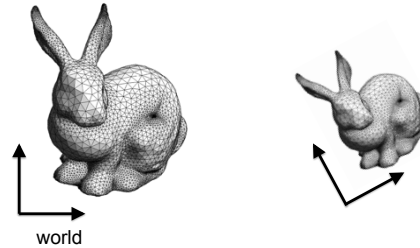
```
glMatrixMode (GL_MODELVIEW);
glLoadIdentity(); // very common usage
float m[16] = { ... };
glLoadMatrixf(m); // rare, advanced
glMultMatrixf(m); // rare, advanced
```

- Use library functions

```
glTranslatef(dx, dy, dz);
glRotatef(angle, vx, vy, vz);
glScalef(sx, sy, sz);
```

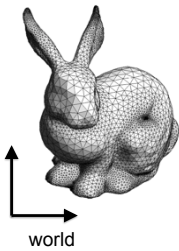
7

Translated, rotated, scaled object



8

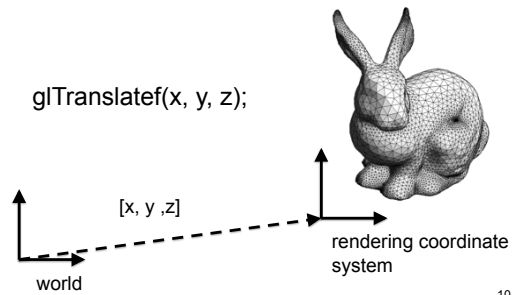
The *rendering* coordinate system



Initially (after `glLoadIdentity()`):
rendering coordinate system =
world coordinate system

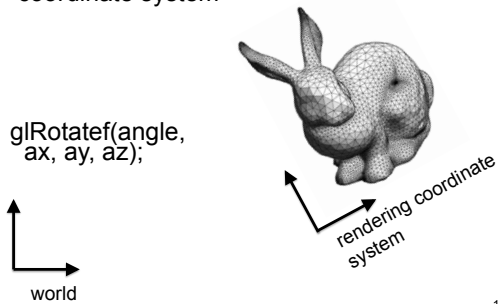
9

The *rendering* coordinate system



10

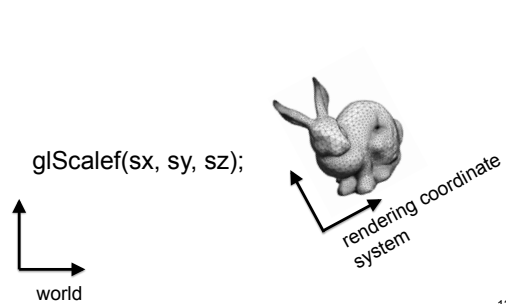
The *rendering* coordinate system



`glRotatef(angle, ax, ay, az);`

11

The *rendering* coordinate system



`glScalef(sx, sy, sz);`

12

OpenGL code

```

glMatrixMode (GL_MODELVIEW);
glLoadIdentity();
glTranslatef(x, y, z);
glRotatef(angle, ax, ay, az);
glScalef(sx, sy, sz);
renderBunny();
    
```

13

Rendering more objects

How to obtain this frame?

14

Solution 1:

Find glTranslate(...), glRotatef(...), glScalef(...)

How to obtain this frame?

15

Solution 2: gl{Push,Pop}Matrix

```

glMatrixMode (GL_MODELVIEW);
glLoadIdentity();

// render first bunny
glPushMatrix(); // store current matrix
glTranslatef(...);
glRotatef(...);
renderBunny();
glPopMatrix(); // pop matrix

// render second bunny
glPushMatrix(); // store current matrix
glTranslatef(...);
glRotatef(...);
renderBunny();
glPopMatrix(); // pop matrix world
    
```

16

3D Math Review

17

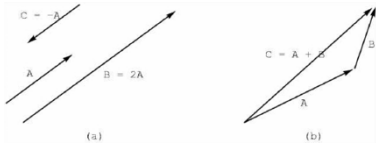
Scalars

- Scalars α, β, γ from a *scalar field*
- Operations $\alpha + \beta, \alpha \cdot \beta, 0, 1, -\alpha, ()^{-1}$
- "Expected" laws apply
- Examples: rationals or reals with addition and multiplication

18

Vectors

- Vectors u, v, w from a *vector space*
- Vector addition $u + v$, subtraction $u - v$
- Zero vector $\mathbf{0}$
- Scalar multiplication αv



19

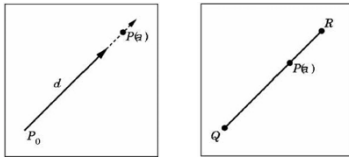
Euclidean Space

- Vector space over real numbers
- Three-dimensional in computer graphics
- Dot product: $\alpha = u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$
- $\mathbf{0} \cdot \mathbf{0} = 0$
- u, v are *orthogonal* if $u \cdot v = 0$
- $|v|^2 = v \cdot v$ defines $|v|$, the *length* of v

20

Lines and Line Segments

- Parametric form of line: $P(\alpha) = P_0 + \alpha d$



- Line segment between Q and R :
 $P(\alpha) = (1-\alpha)Q + \alpha R$ for $0 \leq \alpha \leq 1$

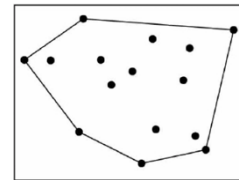
21

Convex Hull

- Convex hull defined by

$$P = \alpha_1 P_1 + \dots + \alpha_n P_n$$

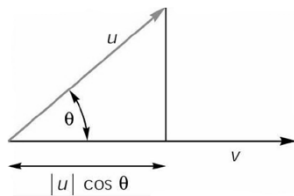
for $\alpha_1 + \dots + \alpha_n = 1$
 and $0 \leq \alpha_i \leq 1, i = 1, \dots, n$



22

Projection

- Dot product projects one vector onto another vector
 $u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3 = |u| |v| \cos(\theta)$
 $pr_v u = (u \cdot v) v / |v|^2$

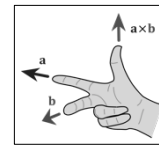
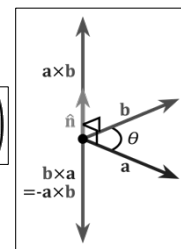


23

Cross Product

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

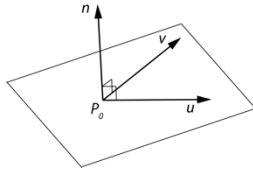
- $|a \times b| = |a| |b| |\sin(\theta)|$
- Cross product is perpendicular to both a and b
- Right-hand rule



Source: Wikipedia

Plane

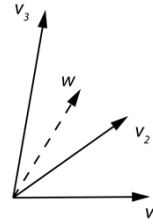
- Plane defined by point P_0 and vectors u and v
- u and v should not be parallel
- Parametric form:
 $T(\alpha, \beta) = P_0 + \alpha u + \beta v$
 (α and β are scalars)
- $n = u \times v / |u \times v|$ is the normal
- $n \cdot (P - P_0) = 0$ if and only if P lies in plane



25

Coordinate Systems

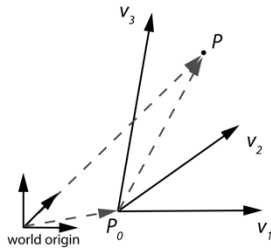
- Let v_1, v_2, v_3 be three linearly independent vectors in a 3-dimensional vector space
- Can write *any* vector w as
 $w = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$
 for some scalars $\alpha_1, \alpha_2, \alpha_3$



26

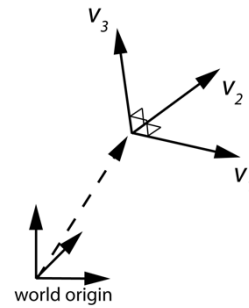
Frames

- Frame = origin P_0 + coordinate system
- Any point $P = P_0 + \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$



27

In Practice, Frames are Often Orthogonal



28

Change of Coordinate System

- Bases $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$
- Express basis vectors u_i in terms of v_j

$$u_1 = \gamma_{11}v_1 + \gamma_{12}v_2 + \gamma_{13}v_3$$

$$u_2 = \gamma_{21}v_1 + \gamma_{22}v_2 + \gamma_{23}v_3$$

$$u_3 = \gamma_{31}v_1 + \gamma_{32}v_2 + \gamma_{33}v_3$$

- Represent in matrix form:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = M \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad M = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix}$$

29

Representing 3D transformations (and model-view matrices)

30

Linear Transformations

- 3 x 3 matrices represent linear transformations
 $\mathbf{a} = \mathbf{M}\mathbf{b}$
- Can represent rotation, scaling, and reflection
- Cannot represent translation

$$M = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix}$$

31

In order to represent rotations, scales AND translations: Homogeneous Coordinates

- Augment $[\alpha_1 \ \alpha_2 \ \alpha_3]^T$ by adding a fourth component (1):

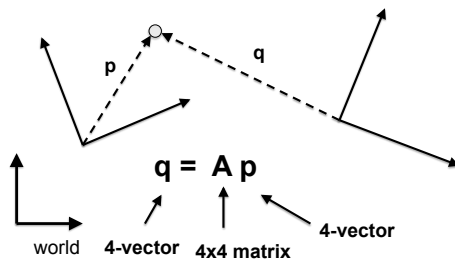
$$\mathbf{p} = [\alpha_1 \ \alpha_2 \ \alpha_3 \ 1]^T$$

- Homogeneous property:

$$\mathbf{p} = [\alpha_1 \ \alpha_2 \ \alpha_3 \ 1]^T = [\beta\alpha_1 \ \beta\alpha_2 \ \beta\alpha_3 \ \beta]^T, \text{ for any scalar } \beta \neq 0$$

32

Homogeneous coordinates are transformed by 4x4 matrices



33

Affine Transformations (4x4 matrices)

- Translation
- Rotation
- Scaling
- Any composition of the above
- Later: projective (perspective) transformations
 - Also expressible as 4 x 4 matrices!

34

Translation

- $\mathbf{q} = \mathbf{p} + \mathbf{d}$ where $\mathbf{d} = [\alpha_x \ \alpha_y \ \alpha_z \ 0]^T$
- $\mathbf{p} = [x \ y \ z \ 1]^T$
- $\mathbf{q} = [x' \ y' \ z' \ 1]^T$
- Express in matrix form $\mathbf{q} = \mathbf{T} \mathbf{p}$ and solve for \mathbf{T}

$$T = \begin{bmatrix} 1 & 0 & 0 & \alpha_x \\ 0 & 1 & 0 & \alpha_y \\ 0 & 0 & 1 & \alpha_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

35

Scaling

- $x' = \beta_x x$
- $y' = \beta_y y$
- $z' = \beta_z z$
- Express as $\mathbf{q} = \mathbf{S} \mathbf{p}$ and solve for \mathbf{S}

$$S = \begin{bmatrix} \beta_x & 0 & 0 & 0 \\ 0 & \beta_y & 0 & 0 \\ 0 & 0 & \beta_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

36

Rotation in 2 Dimensions

- Rotation by θ about the origin
- $x' = x \cos \theta - y \sin \theta$
- $y' = x \sin \theta + y \cos \theta$

- Express in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Note that the determinant is 1

37

Rotation in 3 Dimensions

- Orthogonal matrices:

$$RR^T = R^T R = I \\ \det(R) = 1$$

- Affine transformation:

$$A = \begin{bmatrix} R_{11} & R_{12} & R_{13} & 0 \\ R_{21} & R_{22} & R_{23} & 0 \\ R_{31} & R_{32} & R_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

38

Affine Matrices are Composed by Matrix Multiplication

- $A = A_1 A_2 A_3$
- Applied from right to left
- $A p = (A_1 A_2 A_3) p = A_1 (A_2 (A_3 p))$
- When calling `glTranslate3f`, `glRotatef`, or `glScalef`, OpenGL forms the corresponding 4x4 matrix, and multiplies the current modelview matrix with it.

39

Summary

- OpenGL Transformation Matrices
- Vector Spaces
- Frames
- Homogeneous Coordinates
- Transformation Matrices

40